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Two-mode Integral Form Projection Operator and Its Applications in Quantizing the Mesoscopic Coupling Circuit

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Abstract A ket-bra form quantum operator is constructed and its transformation properties are researched. Furthermore, we quantize two *L*-*C* circuits with coupling and investigate what happens to the two circuits when this system is quantized.

Keywords IWOP technique · Ket-bra form projection operator · Mesoscopic coupled circuit

1 Introduction

With the development of nanometer techniques and microelectronics, the trend in the miniaturization of integrated circuits and electronic components towards atomic scale dimensions becomes strong and definite [1]. Clearly, when the charge-carrier phase coherence length and the charge-carrier confinement dimension reach or approach the Fermi wavelength, the physics of classical devices, based on the motion of particles and ensemble averaging, is expected to be invalid, and quantum effects in electronic devices and circuit must now be taken into account. In the 1970s, the time evolution of an *L*-*C* circuit and its quantum effects were first discussed by Louisell after this system was quantized [2]. Reference [3] studied the quantum effects of coupled double resonance circuit using the scheme of Gauss propagator. Reference [4] studied the quantum fluctuations of mesoscopic damped mutual inductance coupled double resonance RLC circuit at finite temperature. On the other hand, Fan originated the technique of integration within an ordered product (IWOP) of operators [5].

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The IWOP techniques have developed greatly the quantum theory and have been proved to be very useful in studying quantum mechanics, quantum optics and quantum statistics. By virtue of the IWOP techniques, one can create new quantum representations and construct new quantum unitary transformation operators.

In this paper, we construct a ket-bra form quantum operator and research its transformation properties. Furthermore, we quantize two *L-C* circuits with coupling and investigate what happens to the two circuits when this system is quantized.

2 Two-mode Integral Form Projector

Based on the classical transformation $q_1, q_2 \to Aq_1 + Bq_2, Cq_1 + Dq_2$ in the state $|q_1, q_2\rangle$, now an interesting question thus naturally arises: what is the corresponding operator? For this purpose we construct the following ket-bra integration:

$$\hat{U} = \int\int_{-\infty}^{\infty} dq_1 dq_2 \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right|, \tag{1}$$

where A, B, C, D are all real and $AD - BC = 1$. The $\left|\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right\rangle$ is the two-mode coordinate eigenstate, whose expressions are given in two-mode Fock space by

$$\begin{aligned} \left| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle &\equiv |q_1 q_2\rangle \\ &= \pi^{-1/2} \exp\left(-\frac{1}{2}q_1^2 - \frac{1}{2}q_2^2 + \sqrt{2}q_1\hat{a}_1^\dagger + \sqrt{2}q_2\hat{a}_2^\dagger - \frac{1}{2}\hat{a}_1^{\dagger 2} - \frac{1}{2}\hat{a}_2^{\dagger 2} \right) |00\rangle, \end{aligned} \tag{2}$$

$$\begin{aligned} &\left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \\ &= \pi^{-1/2} \exp\left(-\frac{1}{2}(Aq_1 + Bq_2)^2 - \frac{1}{2}(Cq_1 + Dq_2)^2 + \sqrt{2}(Aq_1 + Bq_2)\hat{a}_1^\dagger \right. \\ &\quad \left. + \sqrt{2}(Cq_1 + Dq_2)\hat{a}_2^\dagger - \frac{1}{2}\hat{a}_1^{\dagger 2} - \frac{1}{2}\hat{a}_2^{\dagger 2} \right) |00\rangle. \end{aligned} \tag{3}$$

By virtue of IWOP techniques, the unitarity of $\hat{U}$ can be obtained by calculating

$$\begin{aligned} \hat{U}\hat{U}^\dagger &= \int \cdots \int_{-\infty}^{\infty} dq_1 dq_2 dq_1' dq_2' \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right| \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \right| \\ &= \int\int_{-\infty}^{\infty} dq_1 dq_2 \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right| = 1 = \hat{U}^\dagger \hat{U}. \end{aligned} \tag{4}$$

Using IWOP techniques, we can obtain

$$\begin{aligned} &\hat{U}\hat{q}_1\hat{U}^\dagger \\ &= \int \cdots \int_{-\infty}^{\infty} dq_1 dq_2 dq_1' dq_2' \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right| \hat{q}_1 \left| \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \right| \\ &= \int \cdots \int_{-\infty}^{\infty} q_1 dq_1 dq_2 dq_1' dq_2' \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right| \left| \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \right| \end{aligned}$$

$$= \int \int_{-\infty}^{\infty} q_1 dq_1 dq_2 \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right|$$
$$= D\hat{q}_1 - B\hat{q}_2, \tag{5}$$

where we have used the mathematical formula

$$\int_{-\infty}^{\infty} x e^{-\alpha x^2 + \beta x} dx = \frac{\beta}{2\alpha}\sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha}, \quad \mathrm{Re}(\alpha) > 0.$$

Similarly, we have

$$\hat{U}\hat{q}_2\hat{U}^\dagger = -C\hat{q}_1 + A\hat{q}_2, \quad \hat{U}\hat{p}_1\hat{U}^\dagger = A\hat{p}_1 + C\hat{p}_2, \quad \hat{U}\hat{p}_2\hat{U}^\dagger = B\hat{p}_1 + D\hat{p}_2. \tag{6}$$

From (5) and (6), we know that the unitary transformation, which comes from $\hat{U}$, contains not only squeezing transformation but also rotational transformation. To see this, we perform the integral in (1), i.e.,

$$\hat{U} = \frac{2}{\sqrt{L}} \exp\left\{ \frac{1}{2L} [(A^2 + B^2 - C^2 - D^2)(\hat{a}_1^{\dagger 2} - \hat{a}_2^{\dagger 2}) + 4(AC + BD)\hat{a}_1^\dagger \hat{a}_2^\dagger] \right\}$$
$$\times : \exp\left[(\hat{a}_1^\dagger \quad \hat{a}_2^\dagger)(G - 1) \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \right] :$$
$$\times \exp\left\{ \frac{1}{2L} [(B^2 + D^2 - A^2 - C^2)(\hat{a}_1^2 - \hat{a}_2^2) - 4(AB + CD)\hat{a}_1\hat{a}_2] \right\}, \tag{7}$$

$$L = A^2 + B^2 + C^2 + D^2, G = \frac{2}{L} \begin{pmatrix} A + D & B - C \\ C - B & A + D \end{pmatrix}. \tag{8}$$

This is obtained by using the technique of integration within an ordered product (IWOP) of operators. Equation (7) is the normal product form (denoted by ::) of $\hat{U}$.

3 Quantization of Two Mesoscopic *L-C* Circuits with Coupling

Two *L-C* circuits with copling are drawn in Fig. 1, where L_1 and L_2 are the self-inductance coefficients, C_1 and C_2 are the capacitances, C_0 is the coupling capacitance. The current is $i_\alpha = \dot{q}_\alpha$, $\alpha = 1, 2$, where $q_1 = q_1(t)$ and $q_2 = q_2(t)$ are the electrical charges. If $C_0 = 0$, the circuit system become a constrained system ($q_1 = q_2$). If $C_0 \to \infty$, the circuit system become two independent systems (without coupling). We consider the case of $0 < C_0 < \infty$.

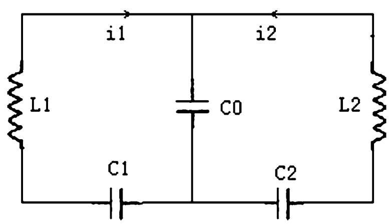

Fig. 1 Two *L-C* circuits with coupling

As can be easily seen, the classical Lagrangian of the system is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2 - \frac{1}{2C_1}q_1^2 - \frac{1}{2C_2}q_2^2 - \frac{1}{2C_0}(q_1+q_2)^2 \\ &= \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2 - \frac{1}{2C_1'}q_1^2 - \frac{1}{2C_2'}q_2^2 - \frac{1}{C_0}q_1q_2,\end{aligned} \tag{9}$$

where $C_1' = C_1C_0/(C_1+C_0)$, $C_2' = C_2C_0/(C_2+C_0)$. From (1) the conjugate momenta p_1 and p_2 are given by

$$p_1 = \frac{\partial\mathcal{L}}{\partial\dot{q}_1} = L_1\dot{q}_1, \quad p_2 = \frac{\partial\mathcal{L}}{\partial\dot{q}_2} = L_2\dot{q}_2, \tag{10}$$

so the variables q_1, q_2 (charge) and $L_1\dot{q}_1, L_2\dot{q}_2$ (magnetic flux) play the role of coordinate and momentum of analytic mechanics. Hence the classical Hamiltonian of the system is $\mathcal{H} = \sum_i p_i\dot{q}_i - \mathcal{L}$. Using the Legendre transformation, we have

$$\mathcal{H} = \frac{p_1^2}{2L_1} + \frac{p_2^2}{2L_2} + \frac{1}{2C_1'}q_1^2 + \frac{1}{2C_2'}q_2^2 + \frac{1}{C_0}q_1q_2, \tag{11}$$

which is analogous to two harmonic oscillators with a coordinate coupling term. We quantize the system by identifying q_α and p_β $(\alpha, \beta = 1, 2)$ as Hermite operators and imposing the commutation relation

$$[\hat{q}_\alpha, \hat{p}_\beta] = i\hbar\delta_{\alpha\beta}, \tag{12}$$

then we obtain the quantum Hamilton operator of the system

$$\hat{\mathcal{H}} = \frac{\hat{p}_1^2}{2L_1} + \frac{\hat{p}_2^2}{2L_2} + \frac{1}{2C_1'}\hat{q}_1^2 + \frac{1}{2C_2'}\hat{q}_2^2 + \frac{1}{C_0}\hat{q}_1\hat{q}_2. \tag{13}$$

In order to diagonalize $\hat{\mathcal{H}}$, we sandwich $\hat{\mathcal{H}}$ between $\hat{U}$ and $\hat{U}^\dagger$, then we have

$$\begin{aligned}\hat{U}\hat{\mathcal{H}}\hat{U}^\dagger &= \left(\frac{A^2}{2L_1} + \frac{B^2}{2L_2}\right)\hat{p}_1^2 + \left(\frac{C^2}{2L_1} + \frac{D^2}{2L_2}\right)\hat{p}_2^2 + \left(\frac{AC}{L_1} + \frac{BD}{L_2}\right)\hat{p}_1\hat{p}_2 \\ &\quad + \left(\frac{D^2}{2C_1'} + \frac{C^2}{2C_2'} - \frac{CD}{C_0}\right)\hat{q}_1^2 + \left(\frac{B^2}{2C_1'} + \frac{A^2}{2C_2'} - \frac{AB}{C_0}\right)\hat{q}_2^2 \\ &\quad + \left(\frac{AD+BC}{C_0} - \frac{BD}{C_1'} - \frac{AC}{C_2'}\right)\hat{q}_1\hat{q}_2.\end{aligned} \tag{14}$$

Without loss of generality, we suppose $L_1C_1'^2 \geq L_2C_2'^2$. Let $A = 1$, $\frac{AC}{L_1} + \frac{BD}{L_2} = 0$, and $\frac{AD+BC}{C_0} - \frac{BD}{C_1'} - \frac{AC}{C_2'} = 0$, therefore we can obtain

$$B = \frac{(L_1C_1' - L_2C_2')C_0 - \sqrt{(L_1C_1' - L_2C_2')^2C_0^2 + 4L_1L_2C_1'^2C_2'^2}}{2L_1C_1'C_2'},$$

$$C = \frac{L_1C_1'C_2'}{\sqrt{4L_1L_2C_1'^2C_2'^2 + C_0^2(L_1C_1' - L_2C_2')^2}},$$

$$D=\frac{1}{2}+\frac{1}{2}\frac{(L_1C_1'-L_2C_2')C_0}{\sqrt{(L_1C_1'-L_2C_2')^2C_0^2+4L_1L_2C_1'^2C_2'^2}}. \tag{15}$$

As a result, $\hat{\mathcal{H}}$ can be diagonalized

$$\hat{U}\hat{\mathcal{H}}\hat{U}^\dagger=\frac{1}{2\mu_1}\hat{p}_1^2+\frac{1}{2\mu_2}\hat{p}_2^2+\frac{1}{2}\mu_1\omega_1^2\hat{q}_1^2+\frac{1}{2}\mu_2\omega_2^2\hat{q}_2^2, \tag{16}$$

where $\mu_1=L_1L_2/(A^2L_2+B^2L_1)$, $\mu_2=L_1L_2/(C^2L_2+D^2L_1)$,

$$\omega_1=\sqrt{\frac{A^2L_2+B^2L_1}{L_1L_2}\left(\frac{D^2}{C_1'}+\frac{C^2}{C_2'}-\frac{2CD}{C_0}\right)},$$

$$\omega_2=\sqrt{\frac{C^2L_2+D^2L_1}{L_1L_2}\left(\frac{B^2}{C_1'}+\frac{A^2}{C_2'}-\frac{2AB}{C_0}\right)}.$$

Sandwiching $\hat{U}\hat{\mathcal{H}}\hat{U}^\dagger$ between $\langle q_1q_2|$ and $\hat{U}|E\rangle$, we have

$$\left(-\frac{\hbar^2}{2\mu_1}\frac{\partial^2}{\partial q_1^2}-\frac{\hbar^2}{2\mu_2}\frac{\partial^2}{\partial q_2^2}+\frac{1}{2}\mu_1\omega_1^2q_1^2+\frac{1}{2}\mu_2\omega_2^2q_2^2\right)\langle q_1q_2|\hat{U}|E\rangle=E\langle q_1q_2|\hat{U}|E\rangle. \tag{17}$$

Obviously, the solutions (eigen-energy levels and eigenfunctions) of (17) are

$$\begin{aligned}&E_{n_1n_2}=(n_1+1/2)\hbar\omega_1+(n_2+1/2)\hbar\omega_2,\\&\langle q_1q_2|\hat{U}|E_{n_1n_2}\rangle=N_{n_1}e^{-\alpha_1^2q_1^2/2}H_{n_1}(\alpha_1q_1)\cdot N_{n_2}e^{-\alpha_2^2q_2^2/2}H_{n_2}(\alpha_2q_2).\end{aligned} \tag{18}$$

Performing $\hat{U}$ on $|q_1q_2\rangle$, we have

$$\hat{U}|q_1q_2\rangle=\left|\begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}q_1\\q_2\end{pmatrix}\right\rangle. \tag{19}$$

So

$$\begin{aligned}\psi_{n_1n_2}(q_1,q_2)&=\langle q_1q_2|E_{n_1n_2}\rangle=\langle q_1q_2|\hat{U}^\dagger\hat{U}|E_{n_1n_2}\rangle=\left\langle\begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}q_1\\q_2\end{pmatrix}\right|\hat{U}|E_{n_1n_2}\rangle\\&=N_{n_1}e^{-\alpha_1^2(Aq_1+Bq_2)^2/2}H_{n_1}[\alpha_1(Aq_1+Bq_2)]\\&\quad\times N_{n_2}e^{-\alpha_2^2(Cq_1+Dq_2)^2/2}H_{n_2}[\alpha_2(Cq_1+Dq_2)].\end{aligned} \tag{20}$$

Hitherto, we have obtained the eigen-wave functions in $\langle q_1q_2|$ representation.

4 Quantum Fluctuations

Using (7), we can obtain

$$\begin{aligned}\hat{U}|00\rangle&=\frac{2}{\sqrt{L}}\exp\left\{\frac{1}{2L}[(A^2+B^2-C^2-D^2)(\hat{a}_1^{\dagger 2}-\hat{a}_2^{\dagger 2})+4(AC+BD)\hat{a}_1^\dagger\hat{a}_2^\dagger]\right\}|00\rangle\\&=\frac{2}{\sqrt{L}}\exp(i\theta\hat{J}_y)\exp\left[\frac{\text{th}r}{2}(\hat{a}_1^{\dagger 2}-\hat{a}_2^{\dagger 2})\right]|00\rangle,\end{aligned} \tag{21}$$

where

$$\hat{J}_y = \frac{1}{2i}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1), \quad \tan\theta = \frac{2(AC+BD)}{C^2+D^2-A^2-B^2},$$
$$\mathrm{th}r = \frac{1}{L}[(A^2+B^2+C^2+D^2)^2 - 4]^{1/2} \tag{22}$$

and we see that $\hat{U}|00\rangle$ denotes a rotated two single-mode squeezed state, $\hat{U}$ includes a squeezing operator. In other words, the quantized two L-C circuits with coupling exhibit squeezing. We now calculate the quantum fluctuations of the state. The averages of charges and their conjugate variables are

$$\bar{q}_1 = \langle 00|\hat{U}^\dagger \hat{q}_1 \hat{U}|00\rangle = 0, \quad \bar{q}_2 = \langle 00|\hat{U}^\dagger \hat{q}_2 \hat{U}|00\rangle = 0,$$
$$\bar{p}_1 = \langle 00|\hat{U}^\dagger \hat{p}_1 \hat{U}|00\rangle = 0, \quad \bar{p}_2 = \langle 00|\hat{U}^\dagger \hat{p}_2 \hat{U}|00\rangle = 0,$$

from which we obtain the variances

$$\overline{(\Delta q_1)^2} = \langle 00|\hat{U}^\dagger \hat{q}_1^2 \hat{U}|00\rangle = (A^2+B^2)/2,$$
$$\overline{(\Delta q_2)^2} = \langle 00|\hat{U}^\dagger \hat{q}_2^2 \hat{U}|00\rangle = (C^2+D^2)/2, \tag{23}$$
$$\overline{(\Delta p_1)^2} = \langle 00|\hat{U}^\dagger \hat{p}_1^2 \hat{U}|00\rangle = (C^2+D^2)/2,$$
$$\overline{(\Delta p_2)^2} = \langle 00|\hat{U}^\dagger \hat{p}_2^2 \hat{U}|00\rangle = (A^2+B^2)/2. \tag{24}$$

Morever, from (10) we have

$$\hat{I}_1 = \hat{\dot{q}}_1 = \frac{\hat{p}_1}{L_1}, \quad \hat{I}_2 = \hat{\dot{q}}_2 = \frac{\hat{p}_2}{L_2}. \tag{25}$$

As a result of (23) and (24) we can obtain the variance of the current in each circuit:

$$\overline{(\Delta I_1)^2} = \frac{1}{L_1^2}\overline{(\Delta p_1)^2} = \frac{C^2+D^2}{2L_1^2}. \tag{26}$$

from (15), (23), (24) and (26) we see that the fluctuations are related to the circuit inherent parameters such as L_1, L_2, C_1, C_2 and C_0. Therefore, studing the quantization and the quantum fluctuations of the mesoscopic coupled circuit is very significant to not only the design of mesoscopic circuits but also for academic guidance.

In conclusion, we have constructed a ket-bra form quantum operator and research its transformation properties. Furthermore, we quantize two L-C circuits with coupling and investigate what happens to the two circuits when this system is quantized.

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